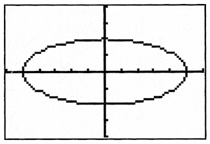
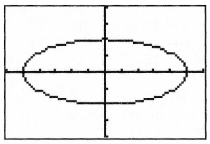
5. $0 \le t \le 2\pi$:



[−6, 6] by [−4, 4]

initial point: (5, 0)
terminal point: (5, 0)

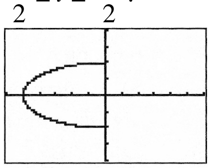
$\pi \le t \le 3\pi$:



[−6, 6] by [−4, 4]

initial point: (−5, 0)
terminal point: (−5, 0)

$\frac{\pi}{2} \le t \le \frac{3\pi}{2}$:



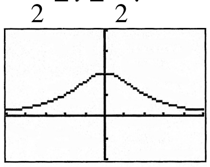
[−6, 6] by [−4, 4]

initial point: (0, −2)
terminal point: (0, 2)
Each curve is traced clockwise from the initial point to the terminal point.

Exploration 3 Graphing the Witch of Agnesi

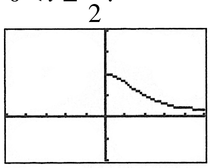
1. We used the parameter interval $[0, \pi]$ because our graphing calculator ignored the fact that the curve is not defined when $t = 0$ or π. The curve is traced from right to left across the screen. x ranges from $-\infty$ to ∞.

2. $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$:



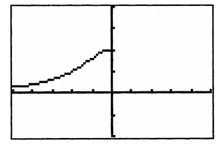
[−5, 5] by [−2, 4]

$0 < t \le \frac{\pi}{2}$:



[−5, 5] by [−2, 4]

$\frac{\pi}{2} \le t < \pi$:



[−5, 5] by [−2, 4]

For $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, the entire graph described in part 1 is drawn. The left branch is drawn from right to left across the screen starting at the point (0, 2). Then the right branch is drawn from right to left across the screen stopping at the point (0, 2). If you leave out $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, then the point (0, 2) is not drawn. For $0 < t \le \frac{\pi}{2}$, the right branch is drawn from right to left across the screen stopping at the point (0, 2). If you leave out $\frac{\pi}{2}$, then the point (0, 2) is not drawn.

For $\frac{\pi}{2} \le t < \pi$, the left branch is drawn from right to left across the screen starting at the point (0, 2). If you leave out $\frac{\pi}{2}$, then the point (0, 2) is not drawn.

3. If you replace $x = 2 \cot t$ by $x = -2 \cot t$, the same graph is drawn except it is traced from left to right across the screen. If you replace $x = 2 \cot t$ by $x = 2 \cot (\pi - t)$, the same graph is drawn except it is traced from left to right across the screen.

Quick Review 1.4

1. $m = \frac{3-8}{4-1} = \frac{-5}{3} = -\frac{5}{3}$

$$y = -\frac{5}{3}(x-1)+8$$

$$y = -\frac{5}{3}x+\frac{29}{3}$$

2. $y = -4$

3. $x = 2$

4. When $y = 0$, we have $\frac{x^2}{9} = 1$, so the x-intercepts are -3 and 3. When $x = 0$, we have $\frac{y^2}{16} = 1$, so the y-intercepts are -4 and 4.

5. When $y = 0$, we have $\frac{x^2}{16} = 1$, so the x-intercepts are -4 and 4. When $x = 0$, we have $-\frac{y^2}{9} = 1$, which has no real solution, so there are no y-intercepts.

6. When $y = 0$, we have $0 = x + 1$, so the x-intercept is -1. When $x = 0$, we have $2y^2 = 1$, so the y-intercepts are $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

7. **(a)** $$\begin{aligned} 2(1)^2(1) + 1^2 &\stackrel{?}{=} 3 \\ 3 &= 3 \quad \text{Yes} \end{aligned}$$

(b) $$\begin{aligned} 2(-1)^2(-1) + (-1)^2 &\stackrel{?}{=} 3 \\ -2 + 1 &\stackrel{?}{=} 3 \\ -1 &\neq 3 \quad \text{No} \end{aligned}$$

(c) $$\begin{aligned} 2\left(\frac{1}{2}\right)^2(-2) + (-2)^2 &\stackrel{?}{=} 3 \\ -1 + 4 &\stackrel{?}{=} 3 \\ 3 &= 3 \quad \text{Yes} \end{aligned}$$

8. **(a)** $$\begin{aligned} 9(1)^2 - 18(1) + 4(3)^2 &= 27 \\ 9 - 18 + 36 &\stackrel{?}{=} 27 \\ 27 &= 27 \quad \text{Yes} \end{aligned}$$

(b) $$\begin{aligned} 9(1)^2 - 18(1) + 4(-3)^2 &\stackrel{?}{=} 27 \\ 9 - 18 + 36 &\stackrel{?}{=} 27 \\ 27 &= 27 \quad \text{Yes} \end{aligned}$$

(c) $$\begin{aligned} 9(-1)^2 - 18(-1) + 4(3)^2 &\stackrel{?}{=} 27 \\ 9 + 18 + 36 &\stackrel{?}{=} 27 \\ 63 &\neq 27 \quad \text{No} \end{aligned}$$

9. **(a)** $$\begin{aligned} 2x + 3t &= -5 \\ 3t &= -2x - 5 \\ t &= \frac{-2x - 5}{3} \end{aligned}$$

(b) $$\begin{aligned} 3y - 2t &= -1 \\ -2t &= -3y - 1 \\ 2t &= 3y + 1 \\ t &= \frac{3y + 1}{2} \end{aligned}$$

10. **(a)** The equation is true for $a \geq 0$.

(b) The equation is equivalent to "$\sqrt{a^2} = a$ or $\sqrt{a^2} = -a$." Since $\sqrt{a^2} = a$ is true for $a \geq 0$ and $\sqrt{a^2} = -a$ is true for $a \leq 0$, at least one of the two equations is true for all real values of a. Therefore, the given equation $\sqrt{a^2} = \pm a$ is true for all real values of a.

(c) The equation is true for all real values of a.

Section 1.4 Exercises

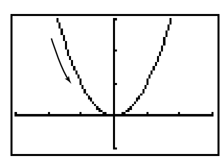
1. Graph (c); window: $[-4, 4]$ by $[-3, 3]$, $0 \leq t \leq 2\pi$

2. Graph (a); window: $[-2, 2]$ by $[-2, 2]$, $0 \leq t \leq 2\pi$

3. Graph (d); window: $[-10, 10]$ by $[-10, 10]$, $0 \leq t \leq 2\pi$

4. Graph (b); window: $[-15, 15]$ by $[-15, 15]$, $0 \leq t \leq 2\pi$

5. **(a)**



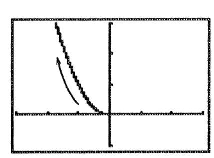
$[-3, 3]$ by $[-1, 3]$

No initial or terminal point

(b) $y = 9t^2 = (3t)^2 = x^2$

The parametrized curve traces all of the parabola defined by $y = x^2$.

6. **(a)**



$[-3, 3]$ by $[-1, 3]$

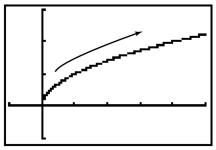
Initial point: (0, 0)

Terminal point: None

(b) $y = t = \left(-\sqrt{t}\right)^2 = x^2$

The parametrized curve traces the left half of the parabola defined by $y = x^2$ (or all of the curve defined by $x = -\sqrt{y}$).

7. (a)



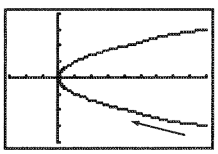
[–1, 5] by [–1, 3]

Initial point: (0, 0)
Terminal point: None

(b) $y = \sqrt{t} = \sqrt{x}$

The parametrized curve traces all of the curve defined by $y = \sqrt{x}$ (or the upper half of the parabola defined by $x = y^2$).

8. (a)



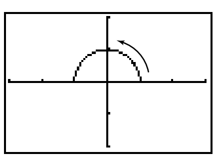
[–3, 9] by [–4, 4]

No initial or terminal point.

(b) $x = \sec^2 t - 1 = \tan^2 t = y^2$

The parametrized curve traces all of the parabola defined by $x = y^2$.

9. (a)



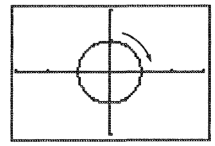
[–3, 3] by [–2, 2]

Initial point: (1, 0)
Terminal point: (–1, 0)

(b) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

The parametrized curve traces the upper half of the circle defined by $x^2 + y^2 = 1$ (or all of the semicircle defined by $y = \sqrt{1 - x^2}$).

10. (a)



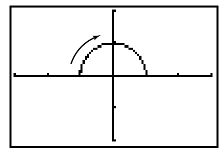
[–3, 3] by [–2, 2]

Initial and terminal point: (0, 1)

(b) $x^2 + y^2 = \sin^2(2\pi t) + \cos^2(2\pi t) = 1$

The parametrized curve traces all of the circle defined by $x^2 + y^2 = 1$.

11. (a)



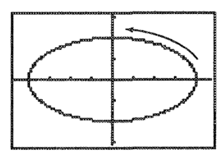
[–3, 3] by [–2, 2]

Initial point: (–1, 0)
Terminal point: (1, 0)

(b) $x^2 + y^2 = \cos^2(\pi - t) + \sin^2(\pi - t) = 1$

The parametrized curve traces the upper half of the circle defined by $x^2 + y^2 = 1$ (or all of the semicircle defined by $y = \sqrt{1 - x^2}$).

12. (a)



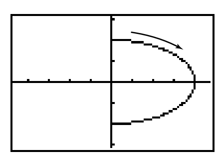
[–4.7, 4.7] by [–3.1, 3.1]

Initial and terminal point: (4, 0)

(b) $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$

The parametrized curve traces all of the ellipse defined by $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

13. (a)



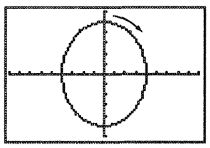
[–4.7, 4.7] by [–3.1, 3.1]

Initial point: (0, 2)
Terminal point: (0, –2)

(b) $\left(\frac{x}{4}\right)^2+\left(\frac{y}{2}\right)^2=\sin^2 t+\cos^2 t=1$

The parametrized curve traces the right half of the ellipse defined by $\left(\frac{x}{4}\right)^2+\left(\frac{y}{2}\right)^2=1$ (or all of the curve defined by $x=2\sqrt{4-y^2}$).

14. **(a)**



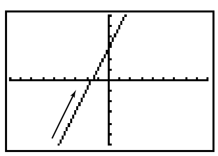
[−9, 9] by [−6, 6]

Initial and terminal point: (0, 5)

(b) $\left(\frac{x}{4}\right)^2+\left(\frac{y}{5}\right)^2=\sin^2 t+\cos^2 t=1$

The parametrized curve traces all of the ellipse defined by $\left(\frac{x}{4}\right)^2+\left(\frac{y}{5}\right)^2=1.$

15. **(a)**



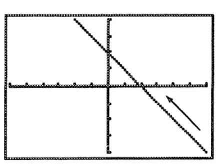
[–9, 9] by [–6, 6]

No initial or terminal point.

(b) $y = 4t - 7 = 2(2t - 5) + 3 = 2x + 3$

The parametrized curve traces all of the line defined by $y = 2x + 3$.

16. **(a)**



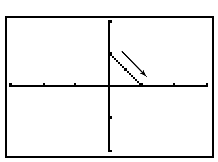
[−6, 6] by [−4, 4]

No initial or terminal point.

(b) $y = 1 + t = 2 - (1 - t) = 2 - x = -x + 2$

The parametrized curve traces all of the line defined by $y = -x + 2$.

17. **(a)**



[–3, 3] by [–2, 2]

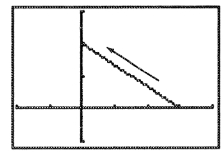
Initial point: (0, 1)

Terminal point: (1, 0)

(b) $y = 1 - t = 1 - x = -x + 1$

The Cartesian equation is $y = -x + 1$. The portion traced by the parametrized curve is the segment from (0, 1) to (1, 0).

18. **(a)**



[−2, 4] by [−1, 3]

Initial point: (3, 0)

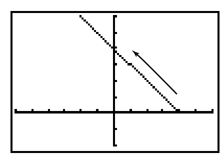
Terminal point: (0, 2)

(b)
$$\begin{aligned} y &= 2t \\ &= (2t-2)+2 \\ &= -\frac{2}{3}(3-3t)+2 \\ &= -\frac{2}{3}x+2 \end{aligned}$$

The Cartesian equation is $y=-\frac{2}{3}x+2.$

The portion traced by the curve is the segment from (3, 0) to (0, 2).

19. **(a)**



[–6, 6] by [–2, 6]

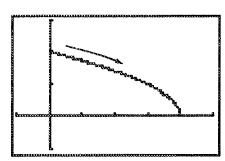
Initial point: (4, 0)

Terminal point: None

(b) $y=\sqrt{t}=4-\left(4-\sqrt{t}\right)=4-x=-x+4$

The parametrized curve traces the portion of the line defined by $y = -x + 4$ to the left of (4, 0), that is, for $x \le 4$.

20. **(a)**



[−1, 5] by [−1, 3]

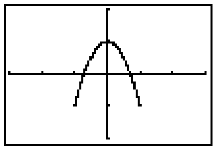
Initial point: (0, 2)

Terminal point: (4, 0)

(b) $y=\sqrt{4-t^2}=\sqrt{4-x}$

The parametrized curve traces the right portion of the curve defined by $y=\sqrt{4-x}$, that is, for $x \ge 0$.

21. (a)



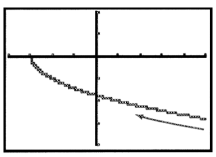
[–3, 3] by [–2, 2]

No initial or terminal point, since the t-interval has no beginning or end. The curve is traced and retraced in both directions.

(b) $y = \cos 2t$
$= \cos^2 t - \sin^2 t$
$= 1 - 2\sin^2 t$
$= 1 - 2x^2$
$= -2x^2 + 1$

The parametrized curve traces the portion of the parabola defined by $y = -2x^2 + 1$ corresponding to $-1 \le x \le 1$.

22. (a)



[−4, 5] by [−4, 2]

Initial point: None
Terminal point: (–3, 0)

(b) $x = t^2 - 3 = y^2 - 3$

The parametrized curve traces the lower half of the parabola defined by $x = y^2 - 3$ (or all of the curve defined by $y = -\sqrt{x+3}$).

23. Using (–1, –3) we create the parametric equations $x = -1 + at$ and $y = -3 + bt$, representing a line which goes through (–1, –3) at $t = 0$. We determine a and b so that the line goes through (4, 1) when $t = 1$.
Since $4 = -1 + a$, $a = 5$.
Since $1 = -3 + b$, $b = 4$.
Therefore, one possible parametrization is $x = -1 + 5t$, $y = -3 + 4t$, $0 \le t \le 1$.

24. Using (–1, 3) we create the parametric equations $x = -1 + at$ and $y = 3 + bt$, representing a line which goes through (–1, 3) at $t = 0$. We determine a and b so that the line goes through (3, –2) at $t = 1$.
Since $3 = -1 + a$, $a = 4$.
Since $-2 = 3 + b$, $b = -5$.
Therefore, one possible parametrization is $x = -1 + 4t$, $y = 3 - 5t$, $0 \le t \le 1$.

25. The lower half of the parabola is given by $x = y^2 + 1$ for $y \le 0$. Substituting t for y, we obtain one possible parametrization: $x = t^2 + 1$, $y = t$, $t \le 0$.

26. The vertex of the parabola is at (–1, –1), so the left half of the parabola is given by $y = x^2 + 2x$ for $x \le -1$. Substituting t for x, we obtain one possible parametrization: $x = t$, $y = t^2 + 2t$, $t \le -1$.

27. For simplicity, we assume that x and y are linear functions of t and that point (x, y) starts at (2, 3) for $t = 0$ and passes through (–1, –1) at $t = 1$. Then $x = 2 + at$ and $y = 3 + bt$.
Since $-1 = 2 + a$, $a = -3$
Since $-1 = 3 + b$, $b = -4$
Therefore, one possible parameterization is $x = 2 - 3t$, $y = 3 - 4t$, $t \ge 0$.

28. For simplicity, we assume that x and y are linear functions of t and that the point (x, y) starts at (–1, 2) for $t = 0$ and passes through (0, 0) at $t = 1$. Then $x = -1 + at$ and $y = 2 + bt$.
Since $0 = -1 + a$, $a = 1$
Since $0 = 2 + b$, $b = -2$
Therefore, one possible parametrization is: $x = -1 + t$, $y = 2 - 2t$, $t \ge 0$.

29. The graph is in Quadrant I when $0 < y < 2$, which corresponds to $1 < t < 3$. To confirm, note that $x(1) = 2$ and $x(3) = 0$.

30. The graph is in Quadrant II when $2 < y \le 4$, which corresponds to $3 < t \le 5$. To confirm, note that $x(3) = 0$ and $x(5) = -2$.

31. The graph is in Quadrant III when $-6 \le y < -4$, which corresponds to $-5 \le t < -3$. To confirm, note that $x(-5) = -2$ and $x(-3) = 0$.

32. The graph is in Quadrant IV when $-4 < y < 0$, which corresponds to $-3 < t < 1$. To confirm, note that $x(-3) = 0$ and $x(1) = 2$.

33. The graph of $y = x^2 + 2x + 2$ lies in Quadrant I for all $x > 0$. Substituting t for x, we obtain one possible parametrization: $x = t$, $y = t^2 + 2t + 2$, $t > 0$.

34. The graph of $y = \sqrt{x+3}$ lies in Quadrant I for all $x > 0$. Substituting t for x, we obtain one possible parametrization: $x = t,\ y = \sqrt{t+3}$, $t > 0$.

35. Possible answers:

(a) $x = a\cos t,\ y = -a\sin t,\ 0 \le t \le 2\pi$

(b) $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$

(c) $x = a\cos t,\ y = -a\sin t,\ 0 \le t \le 4\pi$

(d) $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 4\pi$

36. Possible answers:

(a) $x = -a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$

(b) $x = -a\cos t,\ y = -b\sin t,\ 0 \le t \le 2\pi$

(c) $x = -a\cos t,\ y = b\sin t,\ 0 \le t \le 4\pi$

(d) $x = -a\cos t,\ y = -b\sin t,\ 0 \le t \le 4\pi$

37. False. It is an ellipse.

38. True; circle starting at (2, 0) and ending at (2, 0).

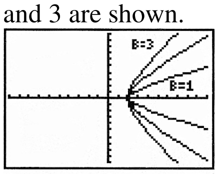
39. D

40. C

41. A

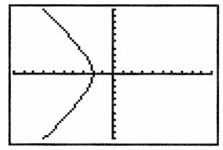
42. E

43. **(a)** The resulting graph appears to be the right half of a hyperbola in the first and fourth quadrants. The parameter a determines the x-intercept. The parameter b determines the shape of the hyperbola. If b is smaller, the graph has less steep slopes and appears “sharper.” If b is larger, the slopes are steeper and the graph appears more “blunt.” The graphs for $a = 2$ and $b = 1, 2,$ and 3 are shown.



[−10, 10] by [−10, 10]

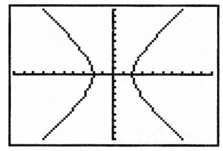
(b)



[−10, 10] by [−10, 10]

This appears to be the left half of the same hyperbola.

(c)



[−10, 10] by [−10, 10]

One must be careful because both sec t and tan t are discontinuous at these points. This might cause the grapher to include extraneous lines (the asymptotes of the hyperbola) in its graph. The extraneous lines can be avoided by using the grapher's dot mode instead of connected mode.

(d) Note that $\sec^2 t - \tan^2 t = 1$ by a standard trigonometric identity. Substituting $\frac{x}{a}$ for sec t and $\frac{y}{b}$ for tan t gives

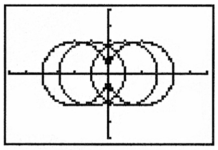
$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

(e) This changes the orientation of the hyperbola. In this case, b determines the y-intercept of the hyperbola, and a determines the shape. The parameter interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ gives the upper half of the hyperbola. The parameter interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ gives the lower half. The same values of t cause discontinuities and may add extraneous lines to the graph.

Substituting $\frac{y}{b}$ for sec t and $\frac{x}{a}$ for tan t in the identity $\sec^2 t - \tan^2 t = 1$ gives

$$\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1.$$

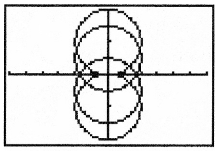
44. (a)



[−6, 6] by [−4, 4]

The graph is a circle of radius 2 centered at $(h, 0)$. As h changes, the graph shifts horizontally.

(b)



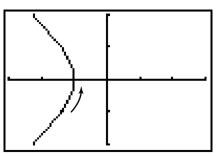
[−6, 6] by [−4, 4]

The graph is a circle of radius 2 centered at $(0, k)$. As k changes, the graph shifts vertically.

(c) Since the circle is to be centered at $(2, -3)$, we use $h = 2$ and $k = -3$. Since a radius of 5 is desired, we need to change the coefficients of $\cos t$ and $\sin t$ to 5.
$x = 5\cos t + 2,\ y = 5\sin t - 3,\ 0 \le t \le 2\pi$

(d) $x = 5\cos t - 3,\ y = 2\sin t + 4,\ 0 \le t \le 2\pi$

45. (a)



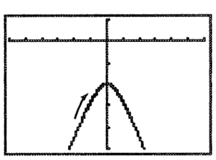
[–3, 3] by [–2, 2]

No initial or terminal point. Note that it may be necessary to use a t-interval such as [–1.57, 1.57] or use dot mode in order to avoid "asymptotes" showing on the calculator screen.

(b) $x^2 - y^2 = \sec^2 t - \tan^2 t = 1$

The parametrized curve traces the left branch of the hyperbola defined by $x^2 - y^2 = 1$ (or all of the curve defined by $x = -\sqrt{y^2+1}$).

46. (a)



[−6, 6] by [−5, 1]

No initial or terminal point. Note that it may be necessary to use a t-interval such as [–1.57, 1.57] or use dot mode in order to avoid "asymptotes" showing on the calculator screen.

(b) $\left(\frac{y}{2}\right)^2 - x^2 = \sec^2 t - \tan^2 t = 1$

The parametrized curve traces the lower branch of the hyperbola defined by $\left(\frac{y}{2}\right)^2 - x^2 = 1$ (or all of the curve defined by $y = -2\sqrt{x^2+1}$)

47. Note that $m\angle OAQ = t$, since alternate interior angles formed by a transversal of parallel lines are congruent. Therefore, $\tan r = \frac{OQ}{AQ} = \frac{2}{x}$, so $x\frac{2}{\tan t} = 2\cot t$. Now, by equation (iii), we know that

$$\begin{aligned} AB &= \frac{(AQ)^2}{AO} \\ &= \left(\frac{AQ}{AO}\right)(AQ) \\ &= (\cos t)(x) \\ &= (\cos t)(2\cot t) \\ &= \frac{2\cos^2 t}{\sin t}. \end{aligned}$$

Then equation (ii) gives
$$\begin{aligned} y &= 2 - AB\sin t \\ &= 2 - \frac{2\cos^2 t}{\sin t}\cdot\sin t \\ &= 2 - 2\cos^2 t \\ &= 2\sin^2 t. \end{aligned}$$

The parametric equations are:
$x = 2\cot t,\ y = 2\sin^2 t,\ 0 < t < \pi$

Note: Equation (iii) may not be immediately obvious, but it may be justified as follows. Sketch segment QB. Then $\angle OBQ$ is a right angle, so $\Delta ABQ \sim \Delta AQO$, which gives $\frac{AB}{AQ} = \frac{AQ}{AO}$.

48. (a) If $x_2 = x_1$ then the line is a vertical line and the first parametric equation gives $x = x_1$, while the second will give all real values for y since it cannot be the case that $y_2 = y_1$ as well. Otherwise, solving the first equation for t gives $t = \frac{(x - x_1)}{(x_2 - x_1)}$.
Substituting that into the second equation gives $y = y_1 + \left[\frac{(y_2 - y_1)}{(x_2 - x_1)}\right](x - x_1)$
which is the point-slope form of the

equation for the line through (x_1, y_1) and (x_2, y_2).
Note that the first equation will cause x to take on all real values, because $(x_2 - x_1)$ is not zero. Therefore, all of the points on the line will be traced out.

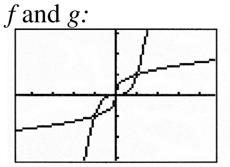
(b) Use the equations for x and y given in part (a), with $0 \le t \le 1$.

Section 1.5 Functions and Logarithms (pp. 36–44)

Exploration 1 Testing for Inverses Graphically

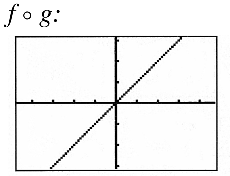
1. It appears that $(f \circ g)(x) = (g \circ f)(x) = x$, suggesting that f and g may be inverses of each other.

(a) f and g:



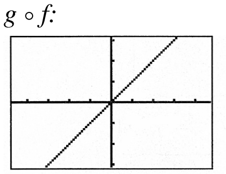
[−4.7, 4.7] by [−3.1, 3.1]

(b) $f \circ g$:



[−4.7, 4.7] by [−3.1, 3.1]

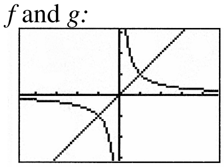
(c) $g \circ f$:



[−4.7, 4.7] by [−3.1, 3.1]

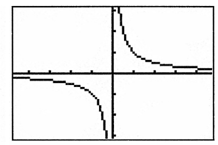
2. It appears that $f \circ g = g \circ f = g$, suggesting that f may be the identity function.

(a) f and g:



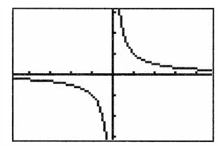
[−4.7, 4.7] by [−3.1, 3.1]

(b) $f \circ g$:



[−4.7, 4.7] by [−3.1, 3.1]

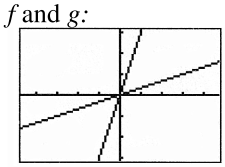
(c) $g \circ f$:



[−4.7, 4.7] by [−3.1, 3.1]

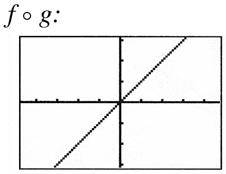
3. It appears that $(f \circ g)(x) = (g \circ f)(x) = x$, suggesting that f and g may be inverses of each other.

(a) f and g:



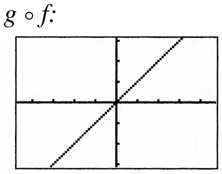
[−4.7, 4.7] by [−3.1, 3.1]

(b) $f \circ g$:



[−4.7, 4.7] by [−3.1, 3.1]

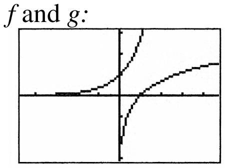
(c) $g \circ f$:



[−4.7, 4.7] by [−3.1, 3.1]

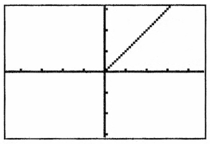
4. It appears that $(f \circ g)(x) = (g \circ f)(x) = x$, suggesting that f and g may be inverses of each other. (Notice that the domain of $f \circ g$ is $(0, \infty)$ and the domain of $g \circ f$ is $(-\infty, \infty)$.)

(a) f and g:



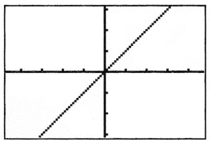
[−4.7, 4.7] by [−3.1, 3.1]

(b) $f \circ g$:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

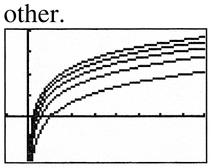
(c) $g \circ f$:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

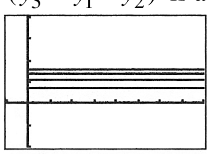
Exploration 2 Supporting the Product Rule

1. They appear to be vertical translations of each other.



$[-1, 8]$ by $[-2, 4]$

2. This graph suggests that each difference $(y_3 = y_1 - y_2)$ is a constant.



$[-1, 8]$ by $[-2, 4]$

3. $$\begin{aligned} y_3 &= y_1 - y_2 \\ &= \ln(ax) - \ln x \\ &= \ln a + \ln x - \ln x \\ &= \ln a \end{aligned}$$
Thus, the difference $y_3 = y_1 - y_2$ is the constant $\ln a$.

Quick Review 1.5

1. $(f \circ g)(1) = f(g(1)) = f(2) = 1$

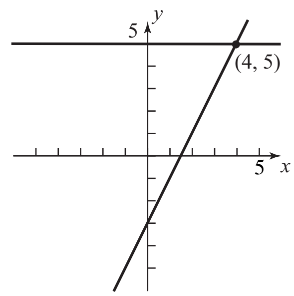
2. $(g \circ f)(-7) = g(f(-7)) = g(-2) = 5$

3. $$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(x^2+1) \\ &= \sqrt[3]{(x^2+1)-1} \\ &= \sqrt[3]{x^2} \\ &= x^{2/3} \end{aligned}$$

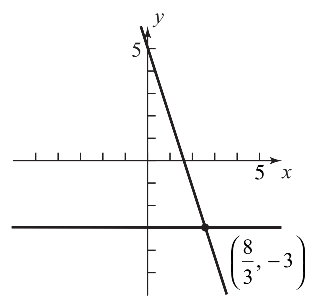
4. $$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g\left(\sqrt[3]{3x-1}\right) \\ &= \left(\sqrt[3]{3x-1}\right)^2 + 1 \\ &= (x-1)^{2/3} + 1 \end{aligned}$$

5. Substituting t for x, one possible answer is:
$x = t,\ y = \dfrac{1}{t-1},\ t \geq 2.$

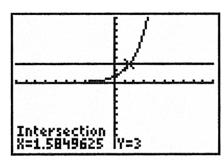
6. Substituting t for x, one possible answer is:
$x = t,\ y = t,\ t < -3$

7. 

(4, 5)

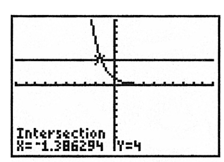
8. 

$\left(\dfrac{8}{3}, -3\right) \approx (2.67, -3)$

9. **(a)** 

$[-10, 10]$ by $[-10, 10]$

(b) No points of intersection, since $2^x > 0$ for all values of x.

10. **(a)** 

$[-10, 10]$ by $[-10, 10]$

$(-1.39, 4)$

(b) No points of intersection, since $e^{-x} > 0$ for all values of x.

Section 1.5 Exercises

1. No, since (for example) the horizontal line $y = 2$ intersects the graph twice.

2. Yes, since each horizontal line intersects the graph only once.

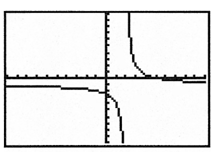
3. Yes, since each horizontal line intersects the graph at most once.

4. No, since (for example) the horizontal line $y = 0.5$ intersects the graph twice.

5. Yes, since each horizontal line intersects the graph only once.

6. No, since (for example) the horizontal line $y = 2$ intersects the graph at more than one point.

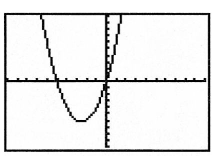
7.



$[-10, 10]$ by $[-10, 10]$

Yes, the function is one-to-one since each horizontal line intersects the graph at most once, so it has an inverse function.

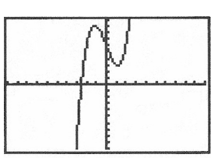
8.



$[-10, 10]$ by $[-10, 10]$

No, the function is not one-to-one since (for example) the horizontal line $y = 0$ intersects the graph twice, so it does not have an inverse function.

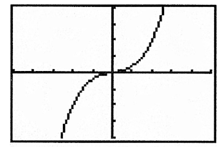
9.



$[-10, 10]$ by $[-10, 10]$

No, the function is not one-to-one since (for example) the horizontal line $y = 5$ intersects the graph more than once, so it does not have an inverse function.

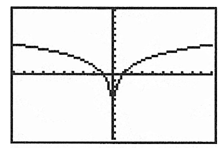
10.



$[-5, 5]$ by $[-20, 20]$

Yes, the function is one-to-one since each horizontal line intersects the graph only once, so it has an inverse function.

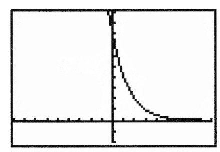
11.



$[-10, 10]$ by $[-10, 10]$

No, the function is not one-to-one since each horizontal line intersects the graph twice, so it does not have an inverse function.

12.



$[-9, 9]$ by $[-2, 10]$

Yes, the function is one-to-one since each horizontal line intersects the graph at most once, so it has an inverse function.

13.

$$y = 2x + 3$$
$$y - 3 = 2x$$
$$\frac{y-3}{2} = x$$

Interchange x and y.

$$\frac{x-3}{2} = y$$
$$f^{-1}(x) = \frac{x-3}{2}$$

Verify.

$$\begin{aligned}(f \circ f^{-1})(x) &= f\left(\frac{x-3}{2}\right) \\ &= 2\left(\frac{x-3}{2}\right) + 3 \\ &= (x-3) + 3 \\ &= x\end{aligned}$$

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(2x+3) \\ &= \frac{(2x+3)-3}{2} \\ &= \frac{2x}{2} \\ &= x\end{aligned}$$

14. $y = 5 - 4x$
$4x = 5 - y$
$x = \dfrac{5-y}{4}$
Interchange x and y.
$y = \dfrac{5-x}{4}$
$f^{-1}(x) = \dfrac{5-x}{4}$
Verify.
$$\begin{aligned}(f \circ f^{-1})(x) &= f\left(\frac{5-x}{4}\right)\\ &= 5 - 4\left(\frac{5-x}{4}\right)\\ &= 5-(5-x)\\ &= x\end{aligned}$$
$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(5-4x)\\ &= \frac{5-(5-4x)}{4}\\ &= \frac{4x}{4}\\ &= x\end{aligned}$$

15. $y = x^3 - 1$
$y+1 = x^3$
$(y+1)^{1/3} = x$
Interchange x and y.
$(x+1)^{1/3} = y$
$f^{-1}(x) = (x+1)^{1/3}$ or $\sqrt[3]{x+1}$
Verify.
$$\begin{aligned}(f \circ f^{-1})(x) &= f\left(\sqrt[3]{x+1}\right)\\ &= \left(\sqrt[3]{x+1}\right)^3 - 1\\ &= (x+1)-1\\ &= x\end{aligned}$$
$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(x^3-1)\\ &= \sqrt[3]{(x^3-1)+1}\\ &= \sqrt[3]{x^3}\\ &= x\end{aligned}$$

16. $y = x^2 + 1,\ x \ge 0$
$y - 1 = x^2,\ x \ge 0$
$\sqrt{y-1} = x$
Interchange x and y.
$\sqrt{x-1} = y$
$f^{-1}(x) = \sqrt{x-1}$ or $(x-1)^{1/2}$
Verify. For $x \ge 1$ (the domain of f^{-1}),
$$\begin{aligned}(f \circ f^{-1})(x) &= f\left(\sqrt{x-1}\right)\\ &= \left(\sqrt{x-1}\right)^2 + 1\\ &= (x-1)+1\\ &= x\end{aligned}$$
For $x > 0$, (the domain of f),
$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(x^2+1)\\ &= \sqrt{(x^2+1)-1}\\ &= \sqrt{x^2}\\ &= |x|\\ &= x\end{aligned}$$

17. $y = x^2,\ x \le 0$
$y = -\sqrt{y}$
Interchange x and y.
$y = -\sqrt{x}$
$f^{-1}(x) = -\sqrt{x}$ or $-x^{1/2}$
Verify.
For $x \ge 0$ (the domain of f^{-1}),
$(f \circ f^{-1})(x) = f\left(-\sqrt{x}\right) = \left(-\sqrt{x}\right)^2 = x$
For $x \le 0$, (the domain of f),
$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(x^2)\\ &= -\sqrt{x^2}\\ &= -|x|\\ &= x\end{aligned}$$

18. $y = x^{2/3},\ x \ge 0$
$y^{3/2} = (x^{2/3})^{3/2},\ x \ge 0$
$y^{3/2} = x$
Interchange x and y.
$x^{3/2} = y$
$f^{-1}(x) = x^{3/2}$
Verify.
For $x \ge 0$ (the domain of f^{-1}),
$(f \circ f^{-1})(x) = f(x^{3/2}) = (x^{3/2})^{2/3} = x$ for
$x \ge 0$, (the domain of f),
$(f^{-1} \circ f)(x) = f^{-1}(x^{2/3}) = (x^{2/3})^{3/2} = |x| = x$

19. $y = -(x-2)^2,\ x \le 2$
$(x-2)^2 = -y,\ x \le 2$
$x - 2 = -\sqrt{-y}$
$x = 2 - \sqrt{-y}$
Interchange x and y.

$$y = 2 - \sqrt{-x}$$
$$f^{-1}(x) = 2 - \sqrt{-x} \text{ or } 2 - (-x)^{1/2}$$

Verify. For $x \le 0$ (the domain of f^{-1}),

$$\begin{aligned}(f \circ f^{-1})(x) &= f\left(2 - \sqrt{-x}\right) \\ &= -\left[\left(2 - \sqrt{-x}\right) - 2\right]^2 \\ &= -\left(-\sqrt{-x}\right)^2 \\ &= -|x| \\ &= x\end{aligned}$$

For $x \le 2$ (the domain of f),

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(-(x-2)^2) \\ &= 2 - \sqrt{(x-2)^2} \\ &= 2 - |x-2| \\ &= 2 + (x-2) \\ &= x\end{aligned}$$

20. $$y = (x^2 + 2x + 1),\ x \ge -1$$
$$y = (x+1)^2,\ x \ge -1$$
$$\sqrt{y} = x + 1$$
$$\sqrt{y} - 1 = x$$

Interchange x and y.

$$\sqrt{x} - 1 = y$$
$$f^{-1}(x) = \sqrt{x} - 1 \text{ or } x^{1/2} - 1$$

Verify. For $x \ge 0$ (the domain of f^{-1}),

$$\begin{aligned}(f \circ f^{-1})(x) &= f\left(\sqrt{x} - 1\right) \\ &= \left[\left(\sqrt{x} - 1\right)^2 + 2\left(\sqrt{x} - 1\right) + 1\right] \\ &= \left(\sqrt{x}\right)^2 - 2\sqrt{x} + 1 + 2\sqrt{x} - 2 + 1 \\ &= \left(\sqrt{x}\right)^2 \\ &= x\end{aligned}$$

For $x \ge -1$ (the domain of f),

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(x^2 + 2x + 1) \\ &= \sqrt{x^2 + 2x + 1} - 1 \\ &= \sqrt{(x+1)^2} - 1 \\ &= |x+1| - 1 \\ &= (x+1) - 1 \\ &= x\end{aligned}$$

21. $$y = \frac{1}{x^2},\ x > 0$$
$$x^2 = \frac{1}{y},\ x > 0$$
$$x = \sqrt{\frac{1}{y}} = \frac{1}{\sqrt{y}}$$

Interchange x and y.

$$y = \frac{1}{\sqrt{x}}$$
$$f^{-1}(x) = \frac{1}{\sqrt{x}} \text{ or } \frac{1}{x^{1/2}}$$

Verify. For $x > 0$ (the domain of f^{-1}),

$$(f \circ f^{-1})(x) = f\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\left(\frac{1}{\sqrt{x}}\right)^2} = x$$

For $x > 0$ (the domain of f),

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}\left(\frac{1}{x^2}\right) \\ &= \frac{1}{\sqrt{\frac{1}{x^2}}} \\ &= \sqrt{x^2} \\ &= |x| \\ &= x\end{aligned}$$

22. $$y = \frac{1}{x^3}$$
$$x^3 = \frac{1}{y}$$
$$x = \sqrt[3]{\frac{1}{y}} = \frac{1}{\sqrt[3]{y}}$$

Interchange x and y.

$$y = \frac{1}{\sqrt[3]{x}}$$
$$f^{-1}(x) = \frac{1}{\sqrt[3]{x}} \text{ or } \frac{1}{x^{1/3}}$$

Verify.

$$(f \circ f^{-1})(x) = f\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{1}{\left(\frac{1}{\sqrt[3]{x}}\right)^3} = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{1}{x^3}\right) = \frac{1}{\sqrt[3]{\frac{1}{x^3}}} = x$$

23. $$y = \frac{2x+1}{x+3}$$
$$xy + 3y = 2x + 1$$
$$xy - 2x = 1 - 3y$$
$$(y-2)x = 1 - 3y$$
$$x = \frac{1-3y}{y-2}$$

Interchange x and y.

$$y = \frac{1-3x}{x-2}$$
$$f^{-1}(x) = \frac{1-3x}{x-2}$$

Verify.

$$(f \circ f^{-1})(x) = f\left(\frac{1-3x}{x-2}\right)$$
$$= \frac{2\left(\frac{1-3x}{x-2}\right)+1}{\frac{1-3x}{x-2}+3}$$
$$= \frac{2(1-3x)+(x-2)}{(1-3x)+3(x-2)}$$
$$= \frac{-5x}{-5}$$
$$= x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{2x+1}{x+3}\right)$$
$$= \frac{1-3\left(\frac{2x+1}{x+3}\right)}{\frac{2x+1}{x+3}-2}$$
$$= \frac{(x+3)-3(2x+1)}{(2x+1)-2(x+3)}$$
$$= \frac{-5x}{-5}$$
$$= x$$

24.
$$y = \frac{x+3}{x-2}$$
$$xy - 2y = x + 3$$
$$xy - x = 2y + 3$$
$$x(y-1) = 2y + 3$$
$$x = \frac{2y+3}{y-1}$$

Interchange x and y.

$$y = \frac{2x+3}{x-1}$$
$$f^{-1}(x) = \frac{2x+3}{x-1}$$

Verify.

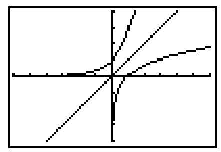
$$(f \circ f^{-1})(x) = f\left(\frac{2x+3}{x-1}\right)$$
$$= \frac{\frac{2x+3}{x-1}+3}{\frac{2x+3}{x-1}-2}$$
$$= \frac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)}$$
$$= \frac{5x}{5}$$
$$= x$$

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{x+3}{x-2}\right)$$
$$= \frac{2\left(\frac{x+3}{x-2}\right)+3}{\frac{x+3}{x-2}-1}$$
$$= \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)}$$
$$= \frac{5x}{5}$$
$$= x$$

25. Graph of f: $x_1 = t,\ y_1 = e^t$

Graph of f^{-1}: $x_2 = e^t,\ y_2 = t$

Graph of $y = x$: $x_3 = t,\ y_3 = t$

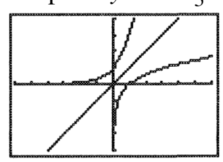


[–6, 6] by [–4, 4]

26. Graph of f: $x_1 = t,\ y_1 = 3^t$

Graph of f^{-1}: $x_2 = 3^t,\ y_2 = t$

Graph of $y = x$: $x_3 = t,\ y_3 = t$

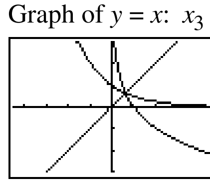


[−6, 6] by [−4, 4]

27. Graph of f: $x_1 = t,\ y_1 = 2^{-t}$

Graph of f^{-1}: $x_2 = 2^{-t},\ y_2 = t$

Graph of $y = x$: $x_3 = t,\ y_3 = t$

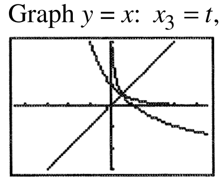


[–4.5, 4.5] by [–3, 3]

28. Graph of f: $x_1 = t,\ y_1 = 3^{-t}$

Graph of f^{-1}: $x_2 = 3^{-t},\ y_2 = t$

Graph $y = x$: $x_3 = t,\ y_3 = t$

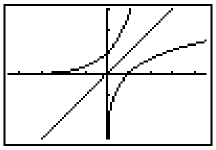


[−4.5, 4.5] by [−3, 3]

29. Graph of f: $x_1 = t,\ y_1 = \ln t$

Graph of f^{-1}: $x_2 = \ln t,\ y_2 = t$

Graph of $y = x$: $x_3 = t,\ y_3 = t$

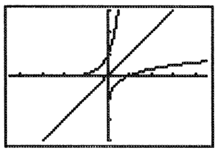


[–4.5, 4.5] by [–3, 3]

30. Graph of f: $x_1 = t,\ y_1 = \log t$

Graph of f^{-1}: $x_2 = \log t,\ y_2 = t$

Graph of $y = x$: $x_3 = t,\ y_3 = t$

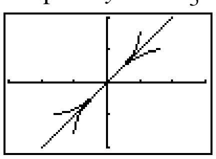


[−4.5, 4.5] by [−3, 3]

31. Graph of f: $x_1 = t,\ y_1 = \sin^{-1} t$

Graph of f^{-1}: $x_2 = \sin^{-1} t,\ y_2 = t$

Graph of $y = x$: $x_3 = t,\ y_3 = t$

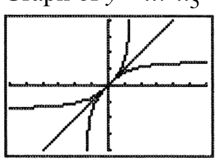


[–3, 3] by [–2, 2]

32. Graph of f: $x_1 = t,\ y_1 = \tan^{-1} t$

Graph of f^{-1}: $x_2 = \tan^{-1} t,\ y_2 = t$

Graph of $y = x$: $x_3 = t,\ y_3 = t$



[−6, 6] by [−4, 4]

33.
$$(1.045)^t = 2$$
$$\ln(1.045)^t = \ln 2$$
$$t \ln 1.045 = \ln 2$$
$$t = \frac{\ln 2}{\ln 1.045} \approx 15.75$$

34.
$$e^{0.05t} = 3$$
$$\ln e^{0.05t} = \ln 3$$
$$0.05t = \ln 3$$
$$t = \frac{\ln 3}{0.05} = 20 \ln 3 \approx 21.97$$

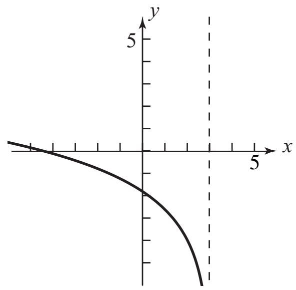
35.
$$e^x + e^{-x} = 3$$
$$e^x - 3 + e^{-x} = 0$$
$$e^x(e^x - 3 + e^{-x}) = e^x(0)$$
$$(e^x)^2 - 3e^x + 1 = 0$$
$$e^x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$
$$e^x = \frac{3 \pm \sqrt{5}}{2}$$
$$x = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right) \approx -0.96 \text{ or } 0.96$$

36.
$$2^x + 2^{-x} = 5$$
$$2^x - 5 + 2^{-x} = 0$$
$$2^x(2^x - 5 + 2^{-x}) = 2^x(0)$$
$$(2^x)^2 - 5(2^x) + 1 = 0$$
$$2^x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)}$$
$$2^x = \frac{5 \pm \sqrt{21}}{2}$$
$$x = \log_2\left(\frac{5 \pm \sqrt{21}}{2}\right) \approx -2.26 \text{ or } 2.26$$

37.
$$\ln y = 2t + 4$$
$$e^{\ln y} = e^{2t+4}$$
$$y = e^{2t+4}$$

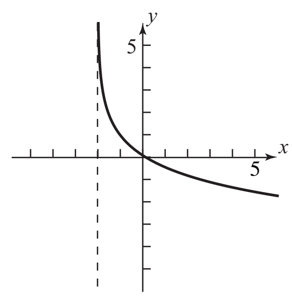
38.
$$\ln(y-1) - \ln 2 = x + \ln x$$
$$\ln(y-1) = x + \ln x + \ln 2$$
$$e^{\ln(y-1)} = e^{x + \ln x + \ln 2}$$
$$y - 1 = e^x(x)(2)$$
$$y = 2xe^x + 1$$

39.



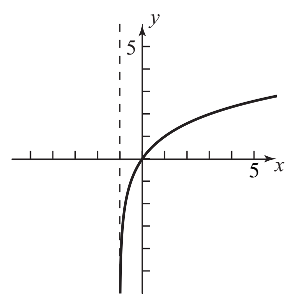
Domain: $(-\infty, 3)$
Range: $(-\infty, \infty)$

40.



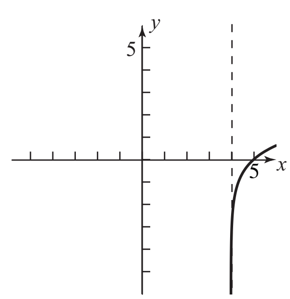
Domain: $(-2, \infty)$
Range: $(-\infty, \infty)$

41.



Domain: $(-1, \infty)$
Range: $(-\infty, \infty)$

42.



Domain: $(4, \infty)$
Range: $(-\infty, \infty)$

43.

$$\begin{aligned} y &= \frac{100}{1+2^{-x}} \\ 1+2^{-x} &= \frac{100}{y} \\ 2^{-x} &= \frac{100}{y}-1 \\ \log_2(2^{-x}) &= \log_2\left(\frac{100}{y}-1\right) \\ -x &= \log_2\left(\frac{100}{y}-1\right) \\ x &= -\log_2\left(\frac{100}{y}-1\right) \\ &= -\log_2\left(\frac{100-y}{y}\right) \\ &= \log_2\left(\frac{y}{100-y}\right) \end{aligned}$$

Interchange x and y.

$$\begin{aligned} y &= \log_2\left(\frac{x}{100-x}\right) \\ f^{-1}(x) &= \log_2\left(\frac{x}{100-x}\right) \end{aligned}$$

Verify.

$$\begin{aligned} (f \circ f^{-1})(x) &= f\left(\log_2 \frac{x}{100-x}\right) \\ &= \frac{100}{1+2^{-\log_2\left(\frac{x}{100-x}\right)}} \\ &= \frac{100}{1+2^{\log_2\left(\frac{100-x}{x}\right)}} \\ &= \frac{100}{1+\frac{100-x}{x}} \\ &= \frac{100x}{x+(100-x)} \\ &= \frac{100x}{100} \\ &= x \end{aligned}$$

$$(f^{-1}\circ f)(x) = f^{-1}\left(\frac{100}{1+2^{-x}}\right)$$
$$= \log_2\left(\frac{\frac{100}{1+2^{-x}}}{100-\frac{100}{1+2^{-x}}}\right)$$
$$= \log_2\left(\frac{100}{100(1+2^{-x})-100}\right)$$
$$= \log_2\left(\frac{1}{2^{-x}}\right)$$
$$= \log_2(2^x)$$
$$= x$$

44.
$$y = \frac{50}{1+1.1^{-x}}$$
$$1+1.1^{-x} = \frac{50}{y}$$
$$1.1^{-x} = \frac{50}{y}-1$$
$$\log_{1.1}(1.1^{-x}) = \log_{1.1}\left(\frac{50}{y}-1\right)$$
$$-x = \log_{1.1}\left(\frac{50}{y}-1\right)$$
$$x = -\log_{1.1}\left(\frac{50}{y}-1\right)$$
$$= -\log_{1.1}\left(\frac{50-y}{y}\right)$$
$$= \log_{1.1}\left(\frac{y}{50-y}\right)$$

Interchange x and y:
$$y = \log_{1.1}\left(\frac{x}{50-x}\right)$$
$$f^{-1}(x) = \log_{1.1}\left(\frac{x}{50-x}\right)$$

Verify.

$$(f\circ f^{-1})(x) = f\left(\log_{1.1}\left(\frac{x}{50-x}\right)\right)$$
$$= \frac{50}{1+1.1^{-\log_{1.1}\left(\frac{x}{50-x}\right)}}$$
$$= \frac{50}{1+1.1^{\log_{1.1}\left(\frac{50-x}{x}\right)}}$$
$$= \frac{50}{1+\frac{50-x}{x}}$$
$$= \frac{50x}{x+(50-x)}$$
$$= \frac{50x}{50}$$
$$= x$$
$$(f^{-1}\circ f)(x) = f^{-1}\left(\frac{50}{1+1.1^{-x}}\right)$$
$$= \log_{1.1}\left(\frac{\frac{50}{1+1.1^{-x}}}{50-\frac{50}{1+1.1^{-x}}}\right)$$
$$= \log_{1.1}\left(\frac{50}{50(1+1.1^{-x})-50}\right)$$
$$= \log_{1.1}\left(\frac{1}{1.1^{-x}}\right)$$
$$= \log_{1.1}(1.1^x)$$
$$= x$$

45. **(a)** $f(f(x)) = \sqrt{1-(f(x))^2}$
$$= \sqrt{1-(1-x^2)}$$
$$= \sqrt{x^2}$$
$$= |x|$$
$$= x, \text{ since } x \ge 0$$

(b) $f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x$ for all $x \ne 0$

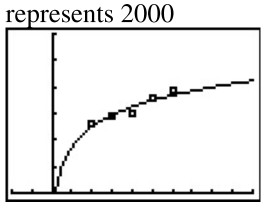
46. **(a)** Amount $= 8\left(\frac{1}{2}\right)^{t/12}$

(b) $$8\left(\frac{1}{2}\right)^{t/12} = 1$$
$$\left(\frac{1}{2}\right)^{t/12} = \frac{1}{8}$$
$$\left(\frac{1}{2}\right)^{t/12} = \left(\frac{1}{2}\right)^{3}$$
$$\frac{t}{12} = 3$$
$$t = 36$$
There will be 1 gram remaining after 36 hours.

47. $$500(1.0475)^t = 1000$$
$$1.0475^t = 2$$
$$\ln(1.0475^t) = \ln 2$$
$$t \ln 1.0475 = \ln 2$$
$$t = \frac{\ln 2}{\ln 1.0475} \approx 14.936$$
It will take about 14.936 years. (If the interest is paid at the end of each year, it will take 15 years.)

48. $$375{,}000(1.0225)^t = 1{,}000{,}000$$
$$1.0225^t = \frac{8}{3}$$
$$\ln(1.0225^t) = \ln\left(\frac{8}{3}\right)$$
$$t \ln 1.0225 = \ln\left(\frac{8}{3}\right)$$
$$t = \frac{\ln\left(\frac{8}{3}\right)}{\ln 1.0225} \approx 44.081$$
It will take about 44.081 years.

49. **(a)** $y = 1.758 + 1.076 \ln(x)$, where $x = 0$ represents 2000

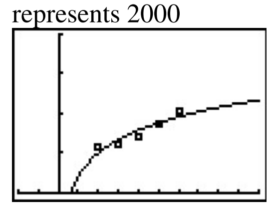


[–2, 10] by [0, 6]

(b) $1.758 + 1.076 \ln(8) \approx 4$
Natural gas production was about 4 trillion cubic feet in 2008.

(c) $$1.758 + 1.076 \ln(x) = 4.2$$
$$1.076 \ln(x) = 2.442$$
$$\ln(x) = \frac{2.442}{1.076}$$
$$x = e^{2.270} \approx 9.68$$
Production was 4.2 trillion cubic feet sometime in 2009.

50. **(a)** $y = 0.434 + 0.8296 \ln(x)$, where $x = 0$ represents 2000



[–2, 10] by [0, 4]

(b) $0.434 + 0.8296 \ln(8) \approx 2.16$
Natural gas production was about 2.16 trillion cubic feet in 2008.

(c) $$0.434 + 0.8296 \ln(x) = 2.25$$
$$0.8296 \ln(x) = 1.816$$
$$\ln(x) = \frac{1.816}{0.8296}$$
$$\ln(x) \approx 2.189$$
$$x = e^{2.189}$$
$$x \approx 8.9$$
Natural gas production was 2.25 trillion cubic feet sometime during 2008.

51. **(a)** Suppose that $f(x_1) = f(x_2)$. Then $mx_1 + b = mx_2 + b$ so $mx_1 = mx_2$. Since $m \neq 0$, this gives $x_1 = x_2$.

(b) $$y = mx + b$$
$$y - b = mx$$
$$\frac{y-b}{m} = x$$
Interchange x and y.
$$\frac{x-b}{m} = y$$
$$f^{-1}(x) = \frac{x-b}{m} = \frac{1}{m}x = \frac{b}{m}$$
The slopes are reciprocals.

(c) If the original functions both have slope m, each of the inverse functions will have slope $\frac{1}{m}$. The graphs of the inverses will be parallel lines with nonzero slope.

(d) If the original functions have slopes m and $-\frac{1}{m}$, respectively, then the inverse functions will have slopes $\frac{1}{m}$ and $-m$, respectively. Since each of $\frac{1}{m}$ and $-m$ is the negative reciprocal of the other, the graphs of the inverses will be perpendicular lines with nonzero slopes.

52. False; for example, the horizontal line test finds three answers for $x = 0$.

53. False; must satisfy $(f \circ g)(x) = (g \circ f)(x) = x,$ not just $(f \circ g)(x) = x.$

54. C; $\ln(x + 2)$ is defined only if $x + 2 > 0$, or $x > -2$.

55. A; the range of the logarithm function is $(-\infty, \infty)$.

56. E;
$$\begin{aligned} f(x) &= 3x-2 \\ y &= 3x-2 \\ 3x &= y+2 \\ x &= \frac{y+2}{3} \end{aligned}$$

Interchange x and y: $y = \frac{x+2}{3}$

57. B;
$$\begin{aligned} 2-3^{-x} &= -1 \\ -3^{-x} &= -3 \\ -3^{-x} &= 3 \\ -x &= 1 \\ x &= -1 \end{aligned}$$

58. **(a)** y_2 is a vertical shift (upward) of y_1, although it's difficult to see that near the vertical asymptote at $x = 0$. One might use "trace" or "table" to verify this.

(b) Each graph of y_3 is a horizontal line.

(c) The graphs of y_4 and $y = a$ are the same.

(d) $e^{y_2 - y_1} = a$, $\ln(e^{y_2-y_1}) = \ln a$, $y_2 - y_1 = \ln a$, $y_1 = y_2 - \ln a = \ln x - \ln a$

59. If the graph of $f(x)$ passes the horizontal line test, so will the graph of $g(x) = -f(x)$ since it's the same graph reflected about the x-axis.
Alternate answer: If $g(x_1) = g(x_2)$ then $-f(x_1) = -f(x_2)$, $f(x_1) = f(x_2)$, and $x_1 = x_2$ since f is one-to-one.

60. Suppose that $g(x_1) = g(x_2)$. Then $\frac{1}{f(x_1)} = \frac{1}{f(x_2)}$, $f(x_1) = f(x_2)$, and x_1 and x_2 since f is one-to-one.

61. **(a)** The expression $a(b^{c-x}) + d$ is defined for all values of x, so the domain is $(-\infty, \infty)$. Since b^{c-x} attains all positive values, the range is (d, ∞) if $a > 0$ and the range is $(-\infty, d)$ if $a < 0$.

(b) The expression $a \log_b(x-c) + d$ is defined when $x - c > 0$, so the domain is (c, ∞). Since $a \log_b(x-c) + d$ attains every real value for some value of x, the range is $(-\infty, \infty)$.

62. **(a)** Suppose $f(x_1) = f(x_2)$. Then:

$$\frac{ax_1 + b}{cx_1 + d} = \frac{ax_2 + b}{cx_2 + d}$$
$$(ax_1 + b)(cx_2 + d) = (ax_2 + b)(cx_1 + d)$$
$$acx_1x_2 + adx_1 + bcx_2 + bd = acx_1x_2 + adx_2 + bcx_1 + bd$$
$$adx_1 + bcx_2 = adx_2 + bcx_1$$
$$(ad - bc)x_1 = (ad - bc)x_2$$

Since $ad - bc \neq 0$, this means that $x_1 = x_2$.

(b)
$$y = \frac{ax + b}{cx + d}$$
$$cxy + dy = ax + b$$
$$(cy - a)x = -dy + b$$
$$x = \frac{-dy + b}{cy - a}$$

Interchanging x and y:

$$y = \frac{-dx + b}{cx - a}$$
$$f^{-1}(x) = \frac{-dx + b}{cx - a}$$

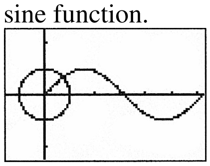
(c) As $x \to \pm\infty$, $f(x) = \dfrac{ax + b}{cx + d} \to \dfrac{a}{c}$, so the horizontal asymptote is $y = \dfrac{a}{c}$ $(c \neq 0)$. Since $f(x)$ is undefined at $x = -\dfrac{d}{c}$, the vertical asymptote is $x = -\dfrac{d}{c}$.

(d) As $x \to \pm\infty$, $f^{-1}(x) = \dfrac{-dx + b}{cx - a} \to -\dfrac{d}{c}$, so the horizontal asymptote is $y = -\dfrac{d}{c}$ $(c \neq 0)$. Since $f^{-1}(x)$ is undefined at $x = \dfrac{a}{c}$, the vertical asymptote is $x = \dfrac{a}{c}$. The horizontal asymptote of f becomes the vertical asymptote of f^{-1} and vice versa due to the reflection of the graph about the line $y = x$.

Section 1.6 Trigonometric Functions (pp. 45–53)

Exploration 1 Unwrapping Trigonometric Functions

1. (x_1, y_1) is the circle of radius 1 centered at the origin (unit circle). (x_2, y_2) is one period of the graph of the sine function.

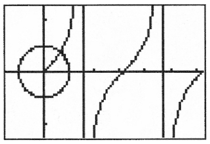


2. The y-values are the same in the interval $0 \leq t \leq 2\pi$.

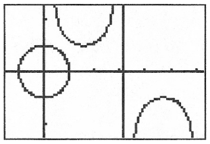
3. The y-values are the same in the interval $0 \leq t \leq 4\pi$.

4. The x_1-values and the y_2-values are the same in each interval.

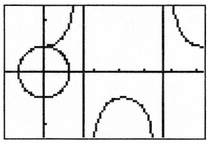
5. $y_2 = \tan t$:



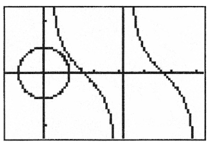
$y_2 = \csc t$:



$y_2 = \sec t$:



$y_2 = \cot t$:



For each value of t, the value of $y_2 = \tan t$ is equal to the ratio $\frac{y_1}{x_1}$.

For each value of t, the value of $y_2 = \csc t$ is equal to the ratio $\frac{1}{y_1}$.

For each value of t, the value of $y_2 = \sec t$ is equal to the ratio $\frac{1}{x_1}$.

For each value of t, the value of $y_2 = \cot t$ is equal to the ratio $\frac{x_1}{y_1}$.

Quick Review 1.6

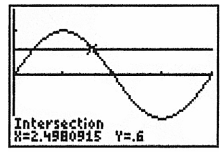
1. $\frac{\pi}{3} \cdot \frac{180°}{\pi} = 60°$

2. $-2.5 \cdot \frac{180°}{\pi} = \left(-\frac{450}{\pi}\right)^{\circ} \approx -143.24°$

3. $-40° \cdot \frac{\pi}{180°} = -\frac{2\pi}{9}$

4. $45° \cdot \frac{\pi}{180°} = \frac{\pi}{4}$

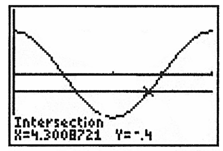
5.



$[0, 2\pi]$ by $[-1.5, 1.5]$

$x \approx 0.6435$, $x \approx 2.4981$

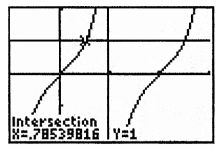
6.



$[0, 2\pi]$ by $[-1.5, 1.5]$

$x \approx 1.9823$, $x \approx 4.3009$

7.



$\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ by $[-2, 2]$

$x \approx 0.7854\left(\text{or } \frac{\pi}{4}\right)$, $x \approx 3.9270 \left(\text{or } \frac{5\pi}{4}\right)$

8. $f(-x) = 2(-x)^2 - 3 = 2x^2 - 3 = f(x)$

The graph is symmetric about the y-axis because if a point (a, b) is on the graph, then so is the point $(-a, b)$.

9.
$$\begin{aligned} f(-x) &= (-x)^3 - 3(-x) \\ &= -x^3 + 3x \\ &= -(x^3 - 3x) \\ &= -f(x) \end{aligned}$$

The graph is symmetric about the origin because if a point (a, b) is on the graph, then so is the point $(-a, -b)$.

10. $x \geq 0$. Alternatively, $x \leq 0$.

Section 1.6 Exercises

1. Arc length $= \left(\frac{5\pi}{8}\right)(2) = \frac{5\pi}{4}$

2. Radius $= \frac{10}{175°\left(\frac{\pi}{180°}\right)} = \frac{72}{7\pi} \approx 3.274$

3. Angle $= \frac{7}{14} = \frac{1}{2}$ radian or about 28.65°.

4. Angle $= \dfrac{\frac{3\pi}{2}}{6} = \dfrac{\pi}{4}$ radian or 45°.

5. Even; $\sec(-\theta) = \dfrac{1}{\cos(-\theta)} = \dfrac{1}{\cos(\theta)} = \sec(\theta)$

6. Odd; $\tan(-\theta) = \dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin(\theta)}{\cos(\theta)} = -\tan\theta$

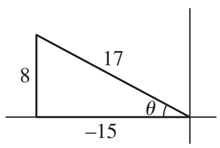
7. Odd; $\csc(-\theta) = \dfrac{1}{\sin(-\theta)} = \dfrac{1}{-\sin(\theta)} = -\csc\theta$

8. Odd; $\cot(-\theta) = \dfrac{\cos(-\theta)}{\sin(-\theta)} = \dfrac{\cos(\theta)}{-\sin(\theta)} = -\cot(\theta)$

9. Using a triangle with sides: −15, 17 and 8;

$\sin\theta = \dfrac{8}{17}$, $\tan\theta = -\dfrac{8}{15}$, $\csc\theta = \dfrac{17}{8}$,

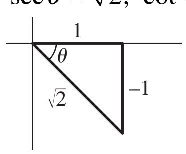
$\sec\theta = -\dfrac{17}{15}$, $\cot\theta = -\dfrac{15}{8}$



10. Using a triangle with sides: −1, 1, and $\sqrt{2}$;

$\sin\theta = -\dfrac{\sqrt{2}}{2}$, $\cos\theta = \dfrac{\sqrt{2}}{2}$, $\csc\theta = -\sqrt{2}$,

$\sec\theta = \sqrt{2}$, $\cot\,\theta = -1$



11. **(a)** Period $= \dfrac{2\pi}{3}$

(b) Domain: Since $\csc(3x+\pi) = \dfrac{1}{\sin(3x+\pi)}$,

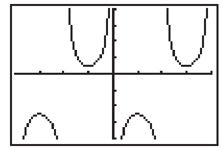
we require $3x + \pi \neq k\pi$, or $x \neq \dfrac{(k-1)\pi}{3}$.

This requirement is equivalent to $x \neq \dfrac{k\pi}{3}$

for integers k.

(c) Since $|\csc(3x + \pi)| \geq 1$, the range excludes numbers between $-3 - 2 = -5$ and $3 - 2 = 1$. The range is $(-\infty, -5] \cup [1, \infty)$.

(d)



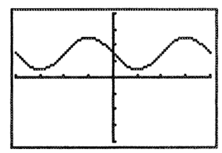
$\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ by $[-8, 8]$

12. **(a)** Period $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

(b) Domain: $(-\infty, \infty)$

(c) Since $|\sin(4x + \pi)| \leq 1$, the range extends from $-2 + 3 = 1$ to $2 + 3 = 5$. The range is $[1, 5]$.

(d)



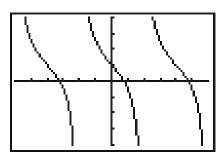
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-8, 8]$

13. **(a)** Period $= \dfrac{\pi}{3}$

(b) Domain: We require $3x + \pi \neq \dfrac{k\pi}{2}$ for odd integers k. Therefore, $x \neq \dfrac{(k-2)\pi}{6}$ for odd integers k. This requirement is equivalent to $x \neq \dfrac{k\pi}{6}$ for odd integers k.

(c) Since the tangent function attains all real values, the range is $(-\infty, \infty)$.

(d)



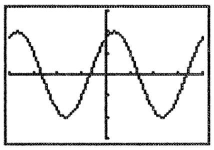
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-8, 8]$

14. **(a)** Period $= \dfrac{2\pi}{2} = \pi$

(b) Domain: $(-\infty, \infty)$

(c) Range: Since $\left|\sin\left(2x+\frac{\pi}{3}\right)\right| \le 1,$ the range is [–2, 2].

(d)



$[-\pi, \pi]$ by $[-3, 3]$

15. (a) The period of $y = \sec x$ is 2π, so the window should have length 4π. One possible answer: $[0, 4\pi]$ by $[-3, 3]$

(b) The period of $y = \csc x$ is 2π, so the window should have length 4π. One possible answer: $[0, 4\pi]$ by $[-3, 3]$

(c) The period of $y = \cot x$ is π, so the window should have length 2π. One possible answer: $[0, 2\pi]$ by $[-3, 3]$

16. (a) The period $y = \sin x$ is 2π, so the window should have length 4π. One possible answer: $[0, 4\pi]$ by $[-2, 2]$

(b) The period of $y = \cos x$ is 2π, so the window should have length 4π. One possible answer: $[0, 4\pi]$ by $[-2, 2]$

(c) The period of $y = \tan x$ is π, so the window should have length 2π. One possible answer: $[0, 2\pi]$ by $[-3, 3]$

17. (a) Period $= \frac{2\pi}{2} = \pi$

(b) Amplitude = 1.5

(c) $[-2\pi, 2\pi]$ by $[-2, 2]$

18. (a) Period $= \frac{2\pi}{3}$

(b) Amplitude = 2

(c) $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ by $[-4, 4]$

19. (a) Period $= \frac{2\pi}{2} = \pi$

(b) Amplitude = 3

(c) $[-2\pi, 2\pi]$ by $[-4, 4]$

20. (a) Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

(b) Amplitude = 5

(c) $[-4\pi, 4\pi]$ by $[-10, 10]$

21. (a) Period $= \frac{2\pi}{\frac{\pi}{3}} = 6$

(b) Amplitude = 4

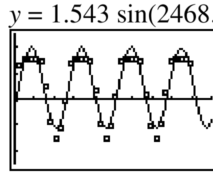
(c) $[-3, 3]$ by $[-5, 5]$

22. (a) Period $= \frac{2\pi}{\pi} = 2$

(b) Amplitude = 1

(c) $[-4, 4]$ by $[-2, 2]$

23. (a) Using a graphing calculator with the sinusoidal regression feature, the equation is $y = 1.543 \sin(2468.635x - 0.494) + 0.438$.



[0, 0.01]by [–2.5, 2.5]

(b) The frequency is 2468.635 radians per second, which is equivalent to $\frac{2468.635}{2\pi} \approx 392.9$ cycles per second (HZ). The note is "G."

24. (a) $b = \frac{2\pi}{12} = \frac{\pi}{6}$

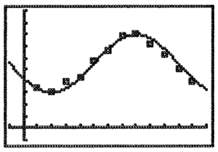
(b) It's half of the difference, so $a = \frac{80-30}{2} = 25.$

(c) $k = \frac{80+30}{2} = 55$

(d) The function should have its minimum at $t = 2$ (when the temperature is 30°F) and its maximum at $t = 8$ (when the temperature is 80°F). The value of h is

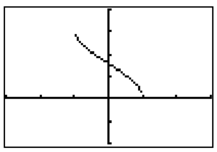
$\frac{2+8}{2} = 5.$ Equation:

$y = 25\sin\left[\frac{\pi}{6}(x-5)\right]+55$

(e) 

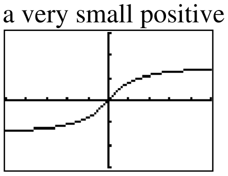
[−1, 13] by [−10, 100]

25. The portion of the curve $y = \cos x$ between $0 \le x \le \pi$ passes the horizontal line test, so it is one-to-one.



[–3, 3] by [–2, 4]

26. The portion of the curve $y = \tan x$ between $-\frac{\pi}{2} < x < \frac{\pi}{2}$ passes the horizontal line test, so it is one-to-one. [In parametric mode, use $T_{\min} = -\frac{\pi}{2} + \varepsilon$ and $T_{\max} = \frac{\pi}{2} - \varepsilon$, where ε is a very small positive number, say 0.00001.]



[–5, 5] by [–3, 3]

27. Since $\frac{\pi}{6}$ is in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or

$y = \sin^{-1} x$ and $\sin\frac{\pi}{6} = 0.5,$

$\sin^{-1}(0.5) = \frac{\pi}{6}$ radian or $\frac{\pi}{6}\cdot\frac{180°}{\pi} = 30°.$

28. Since $-\frac{\pi}{4}$ is in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of

$y = \sin^{-1} x$ and $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2},$

$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ radian or

$-\frac{\pi}{4}\cdot\frac{180°}{\pi} = -45°.$

29. Using a calculator,
$\tan^{-1}(-5) \approx -1.3734$ radians or $-78.6901°$.

30. Using a calculator, $\cos^{-1}(0.7) = 0.7954$ radian or $45.5730°$.

31. The angle $\tan^{-1}(2.5) \approx 1.190$ is the solution to this equation in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}.$ Another solution in $0 \le x \le 2\pi$ is $\tan^{-1}(2.5) + \pi \approx 4.332.$ The solutions are $x \approx 1.190$ and $x \approx 4.332$.

32. The angle $\cos^{-1}(-0.7) \approx 2.346$ is the solution to this equation in the interval $0 \le x \le \pi$. Since the cosine function is even, the value $-\cos^{-1}(-0.7) \approx -2.346$ is also a solution, so any value of the form $\pm\cos^{-1}(-0.7) + 2k\pi$ is a solution, where k is an integer. In $2\pi \le x < 4\pi$ the solutions are $x = \cos^{-1}(-0.7) + 2\pi \approx 8.629$ and $x = -\cos^{-1}(-0.7) + 4\pi \approx 10.200.$

33. The equation is equivalent to $\sin x = \frac{1}{2},$ so the solutions in the interval $0 \le x < 2\pi$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}.$

34. This equation is equivalent to $\cos x = -\frac{1}{3},$ so the solution in the interval $0 \le x \le \pi$ is $y = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.911.$ Since the cosine function is even, the solutions in the interval $-\pi \le x < \pi$ are $x \approx -1.911$ and $x \approx 1.911$.

35. The solutions in interval $0 \le x < 2\pi$ are $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}.$ Since $y = \sin x$ has period 2π, the solutions are all of the form $x = \frac{7\pi}{6} + 2k\pi$ or $x = \frac{11\pi}{6} + 2k\pi,$ where k is any integer.

36. The equation is equivalent to $\tan x = \frac{1}{-1} = -1$, to the solution in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is $x = \tan^{-1}(-1) = -\frac{\pi}{4}$. Since the period of $y = \tan x$ is π, all solutions are of the form $x = -\frac{\pi}{4} = k\pi$, where k is any integer. This is equivalent to $x = \frac{3\pi}{4} + k\pi$, where k is any integer.

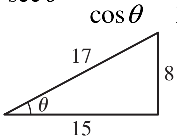
37. Note that $\sqrt{8^2 + 15^2} = 17$. Since $\sin\theta = \frac{8}{17}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$,

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \frac{15}{17}.$$

Therefore: $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{8}{15}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{15}{8},$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{17}{15}, \quad \csc\theta = \frac{1}{\sin\theta} = \frac{17}{8}$$



38. Note that $\sqrt{5^2 + 12^2} = 13$.

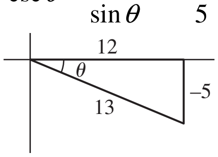
Since $\tan\theta = -\frac{5}{12} = \frac{-\frac{5}{13}}{\frac{12}{13}} = \frac{\sin\theta}{\cos\theta}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we have $\sin\theta = -\frac{5}{13}$ and $\cos\theta = \frac{12}{13}$.

In summary:

$$\sin\theta = -\frac{5}{13}, \cos\theta = \frac{12}{13}, \tan\theta = -\frac{5}{12}$$

$$\cot\theta = \frac{1}{\tan\theta} = -\frac{12}{5}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{13}{12},$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{13}{5}$$

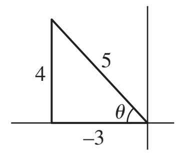


39. Note that $r = \sqrt{(-3)^2 + 4^2} = 5$. Then:

$$\sin\theta = \frac{y}{r} = \frac{4}{5}, \cos\theta = \frac{x}{r} = -\frac{3}{5},$$

$$\tan\theta = \frac{y}{x} = -\frac{4}{3}, \quad \cot\theta = \frac{x}{y} = -\frac{3}{4},$$

$$\sec\theta = \frac{r}{x} = -\frac{5}{3}, \quad \csc\theta = \frac{r}{y} = \frac{5}{4}$$



40. Note that $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$. Then:

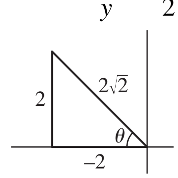
$$\sin\theta = \frac{y}{r} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}},$$

$$\tan\theta = \frac{y}{x} = \frac{2}{-2} = -1, \quad \cot\theta = \frac{x}{y} = \frac{-2}{2} = -1,$$

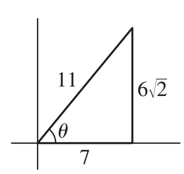
$$\sec\theta = \frac{r}{x} = \frac{2\sqrt{2}}{-2} = -\sqrt{2},$$

$$\csc\theta = \frac{r}{y} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$



41. Let $\theta = \cos^{-1}\left(\frac{7}{11}\right)$. Then $0 \le \theta \le \pi$ and $\cos\theta = \frac{7}{11}$, so

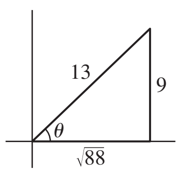
$$\sin\left(\cos^{-1}\left(\frac{7}{11}\right)\right) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{7}{11}\right)^2} = \frac{\sqrt{72}}{11} = \frac{6\sqrt{2}}{11} \approx 0.771$$



42. Let $\theta = \sin^{-1}\left(\frac{9}{13}\right)$. Then $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\sin\theta = \frac{9}{13}$, so

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{9}{13}\right)^2} = \frac{\sqrt{88}}{13}.$$

Therefore,
$$\tan\left(\sin^{-1}\left(\frac{9}{13}\right)\right) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{9}{13}}{\frac{\sqrt{88}}{13}} = \frac{9}{\sqrt{88}} \approx 0.959.$$



43. **(a)**
$$\text{Amplitude} = \frac{\text{high point} - \text{low point}}{2} = \frac{(62-(-12))}{2} = \frac{74}{2} = 37$$

(b) Period = 1 year = 365 days

(c) The graph crosses the green line around April 11 which is the 101st day of the year. Horizontal shift = 101

(d) Vertical shift = height of green line = 25

(e) $f(x) = 37\sin\left[\frac{2\pi}{365}(x-101)\right] + 25$

44. **(a)** Highest: 25 + 37 = 62°F
Lowest: 25 − 37 = −12°F

(b) $\text{Average} = \frac{62+(-12)}{2} = 25°\text{F}$
This average is the same as the vertical shift because the average of the highest and lowest values of $y = \sin x$ is 0.

45. **(a)** $\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos(x)}{-\sin(x)} = -\cot(x)$

(b) Assume that f is even and g is odd. Then $\frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -\frac{f(x)}{g(x)}$ so $\frac{f}{g}$ is odd. The situation is similar for $\frac{g}{f}$.

46. **(a)** $\csc(-x) = \frac{1}{\sin(-x)} = \frac{1}{-\sin(x)} = -\csc(x)$

(b) Assume that f is odd. Then $\frac{1}{f(-x)} = \frac{1}{-f(x)} = -\frac{1}{f(x)}$ so $\frac{1}{f}$ is odd.

47. Assume that f is even and g is odd. Then $f(-x)g(-x) = (f(x))(-g(x)) = -f(x)g(x)$ so fg is odd.

48. If (a, b) is the point on the unit circle corresponding to the angle θ, then $(-a, -b)$ is the point on the unit circle corresponding to the angle $(\theta + \pi)$ since it is exactly half way around the circle. This means that both $\tan(\theta)$ and $\tan(\theta + \pi)$ have the same value, $\frac{b}{a}$.

49. **(a)** Using a graphing calculator with the sinusoidal regression feature, the equation is
$y = 3.0014 \sin(0.9996x + 2.0012) + 2.9999$.

(b) $y = 3 \sin(x + 2) + 3$

50. False; it is 4π because $\frac{2\pi}{B} = \frac{1}{2}$ implies the period B is 4π.

51. False; the amplitude is $\frac{1}{2}$.

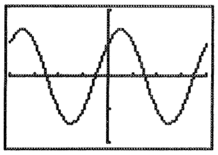
52. D

53. B; the curve oscillates between −3 and 1.

54. E

55. A

56. **(a)**



$[-2\pi, 2\pi]$ by $[-2, 2]$

The graph is a sine/cosine type graph, but it is shifted and has an amplitude greater than 1.

(b) Amplitude ≈ 1.414 $\left(\text{that is, } \sqrt{2}\right)$

Period $= 2\pi$

Horizontal shift $= -0.785$ $\left(\text{that is, } -\dfrac{\pi}{4}\right)$ or 5.498 $\left(\text{that is, } \dfrac{7\pi}{4}\right)$

Vertical shift $= 0$

(c) $$\sin\left(x+\frac{\pi}{4}\right) = (\sin x)\left(\cos\frac{\pi}{4}\right) + (\cos x)\left(\sin\frac{\pi}{4}\right)$$
$$= (\sin x)\left(\frac{1}{\sqrt{2}}\right) + (\cos x)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}(\sin x + \cos x)$$

Therefore, $\sin x + \cos x = \sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)$.

57. **(a)** $\sqrt{2}\sin\left(ax+\dfrac{\pi}{4}\right)$

(b) See part (a).

(c) It works.

(d) $$\sin\left(ax+\frac{\pi}{4}\right) = (\sin ax)\left(\cos\frac{\pi}{4}\right) + (\cos ax)\left(\sin\frac{\pi}{4}\right)$$
$$= (\sin ax)\left(\frac{1}{\sqrt{2}}\right) + (\cos ax)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}(\sin ax + \cos ax)$$

So, $\sin(ax) + \cos(ax) = \sqrt{2}\sin\left(ax+\dfrac{\pi}{4}\right)$.

58. **(a)** One possible answer: $y = \sqrt{a^2+b^2}\,\sin\left(x + \tan^{-1}\left(\dfrac{b}{a}\right)\right)$

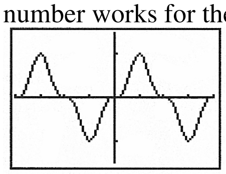
(b) See part (a).

(c) It works.

(d) $$\sin\left(x+\tan^{-1}\left(\frac{b}{a}\right)\right)=\sin(x)\cos\left(\tan^{-1}\left(\frac{b}{a}\right)\right)+\cos(x)\sin\left(\tan^{-1}\left(\frac{b}{a}\right)\right)$$
$$=\sin(x)\left(\frac{a}{\sqrt{a^2+b^2}}\right)+\cos(x)\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$
$$=\frac{1}{\sqrt{a^2+b^2}}\cdot(a\sin x+b\cos x)$$

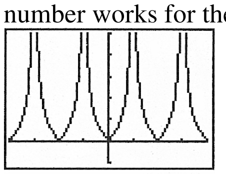
and multiplying through by the square root gives the desired result. Note that the substitutions $\cos\left(\tan^{-1}\frac{b}{a}\right)=\frac{a}{\sqrt{a^2+b^2}}$ and $\sin\left(\tan^{-1}\frac{b}{a}\right)=\frac{b}{\sqrt{a^2+b^2}}$ depend on the requirement that a is positive. If a is negative, the formula does not work.

59. Since $\sin x$ has period 2π, $\sin^3(x+2\pi)=\sin^3(x)$. This function has period 2π. A graph shows that no smaller number works for the period.



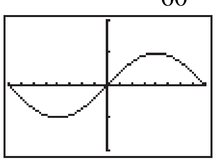
$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

60. Since $\tan x$ has period π, $|\tan(x+\pi)| = |\tan x|$. This function has period π. A graph shows that no smaller number works for the period.



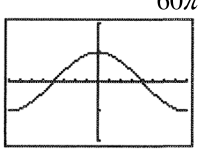
$[-2\pi, 2\pi]$ by $[-1, 5]$

61. The period is $\frac{2\pi}{60}=\frac{\pi}{30}$. One possible graph:



$\left[-\frac{\pi}{60}, \frac{\pi}{60}\right]$ by $[-2, 2]$

62. The period is $\frac{2\pi}{60\pi}=\frac{1}{30}$. One possible graph:



$\left[-\frac{1}{60}, \frac{1}{60}\right]$ by $[-2, 2]$

Quick Quiz (Sections 1.4–1.6)

1. C

2. D